# ON FINITE STRAIN-DISPLACEMENT RELATIONS

By

William C. Orthwein

Space Sciences Division
NASA, Ames Research Center
Moffett Field, Calif.



1. Introduction. Apparently the Green-St. Venant and the Alamansi-Hamel strain tensors were first expressed in curvilinear coordinates in terms of displacements as

$$E_{AB} = \frac{1}{2}\left(u_{A|B} + u_{B|A} + u_{C|A}\, u^{C}{}_{|B}\right) \tag{1}$$

and as

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i|j} + u_{j|i} - u_{k|i}\, u^{k}{}_{|j}\right) \tag{2}$$

by Green and Zerna [1]. Although the Cartesian counterparts of (1) and (2) had been enunciated much earlier, their extension to curvilinear reference frames has been found to be fraught with difficulties. Since it is well known that a satisfactory solution to many problems in continuum mechanics can be achieved only through the use of curvilinear coordinates, these difficulties must be surmounted.

Forms (1) and (2) were later repeated in a text by Green and Zerna [2], and were rederived by Doyle and Ericksen [3]. They have also played an explicit role in the works of Green and Adkins [4], Eringen [5], and others. In all of these $E_{AB}$ represents ~~the measure~~ of strain tensor written in terms of a coordinate system used to describe the undeformed configuration. In this system the deformation is represented by $U^A$, or $U_A$. This same deformation may be measured with reference to a coordinate system chosen to describe the deformed body, in which case the strain is recorded in terms of $\epsilon_{ij}$ and the displacement is represented by $u^i$, or $u_i$. Strain in both reference systems has been related to the covariant derivatives $U_{A|B}$ or $u_{i|j}$ as shown.

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It is the purpose of this paper to show (i) that the use of covariant derivatives of the displacement is incorrect, and (ii) that the correct form of (1) and (2) involves only ordinary partial derivatives of the displacements.

2. Nontensor Character of $U_{A|B}$ and $u_{i|j}$. All of the above authors define $U_{A|B}$ and $u_{i|j}$ as

$$U_{A|B} = U_{A,B} - \Gamma^{C}_{AB} U_C \tag{3}$$

and

$$u_{i|j} = u_{i,j} - \Gamma^{k}_{ij} u_k \tag{4}$$

in which the comma denotes partial differentiation with respect to the coordinates whose indices follow, and where $\Gamma^{i}_{jk}$ and $\Gamma^{A}_{BC}$ denote Christoffel symbols of the second kind. Meaningless though it may be, the formal operations appearing on the right-hand side of (3) and (4) can be applied to nontensors as well as to tensors. However, when these operations are applied to a tensor $t_i$ (or $T_A$), the theory guarantees that $t_{i|j}$ (or $T_{A|B}$) is likewise a tensor. When they are applied to a nontensor, the theory does not supply such a guarantee.

The cornerstone of the present analysis is the observation that large displacements are not tensors and hence that $U_{A|B}$ and $u_{i|j}$ are not tensors. Consider, for example, a displacement $u_2$ in a circular cylindrical coordinate system $x^i$, as shown in Fig. 1, where $z^\alpha$ is taken as a Cartesian frame. If $u_i$ and $u_\alpha$ are tensors, then

$$U_\alpha = x^i_\alpha u_i \tag{5}$$

should hold, where

$$x^i_\alpha \equiv \frac{\partial x^i}{\partial z^\alpha} . \tag{6}$$

According to Fig. 1, in which $x^3$ and $z^3$ are perpendicular to the plane of the page,

$$\left.\begin{aligned} x^1 &= \left[(z^1)^2 + (z^2)^2\right]^{1/2} \\ x^2 &= \tan^{-1}(z^2/z^1) \\ x^3 &= z^3 \end{aligned}\right\} \tag{7}$$

and the displacements in the two systems are related according to

$$\left.\begin{aligned} U_1 &= x^1[\cos(x^2 + u^2) - \cos x^2] + u^1 \cos(x^2 + u^2) \\ U_2 &= x^1[\sin(x^2 + u^2) - \sin x^2] + u^1 \sin(x^2 + u^2) \\ U_3 &= u^3 \cdot , \end{aligned}\right\} \quad (8)$$

wherein $U_\alpha$ refers to the Cartesian frame and $u^i$ refers to the circular cylindrical frame. Carrying out the summations implied in (5), in which the $x^i_\alpha$ are determined from (7), we find that (5) becomes

$$\left.\begin{aligned} U_1 &= u_1 \cos x^2 - (u_2/x^1)\sin x^2 \\ U_2 &= u_1 \sin x^2 + (u_2/x^1)\cos x^2 \\ U_3 &= u_3 \ . \end{aligned}\right\} \quad (9)$$

When $(u^2)^2 \ll 1$ equation (8) becomes

$$\left.\begin{aligned} U_1 &= -x^1u^2 \sin x^1 + u_1 \cos x^2 \\ U_2 &= x^1u^2 \cos x^2 + u^1 \sin x^2 \\ U_3 &= 0 \ , \end{aligned}\right\} \quad (10)$$

which may be seen to agree with (9) when we recall that

$$u_2 = (x^1)^2u^2$$
$$u_1 = u^1 \ .$$

Thus it is evident in the example that (5) holds only in the limiting case of infinitesimal displacements. Conversely, $U_A$ and $u_i$ are not tensors when their magnitude is unrestricted, and thus the presumption that $U_{A|B}$ and $u_{i|j}$ possess the tensor character implied by their indices is unjustified. Stated in more general terms, (1) and (2) are valid only when (i) the displacements are infinitesimal, or (ii) when the displacement is radial as measured in either a spherical or circular cylindrical coordinate system.

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If $U_A$ and $u_i$ were tensors, regardless of their magnitudes, it would then be correct to write that

$$U_{A|B} = Z^{\alpha}_{A} Z^{\beta}_{B} U_{\alpha,\beta} \tag{11}$$

and

$$u_{i|j} = z^{\alpha}_{i} z^{\beta}_{j} u_{\alpha,\beta} . \tag{12}$$

Since neither $U_A$ nor $u_i$ is a tensor, however, neither (11) nor (12) is correct. Continuing with the use of circular cylindrical coordinates, for example, the left-hand side of (12) yields

$$u_{1|1} = u_{1,1} = u^1{}_{,1} \tag{13}$$

but the right-hand side is such that

$$z^{\alpha}_{1} z^{\beta}_{1} u_{\alpha,\beta} = (u^1{}_{,1} + u^2{}_{,1} + 1)\cos u^2 - 1 \tag{14}$$

Should the displacement be only radial, then $u^2 \equiv 0$ and the right-hand sides of (13) and (14) agree. This is indicative of the behavior of other components of (11) and (12). Since direct calculation shows that (11) and (12) fail to hold, it is again evident that the quantities $U_{A|B}$ and $u_{i|j}$ do not have general tensor character, a conclusion established by the earlier failure of (5).

3. Derivation of Strain-Displacement Relations. It will prove convenient to consider six coordinate systems, three for the initial configuration of the body and three for the deformed configuration. In the initial configuration the material points are assigned coordinates $Z^{\alpha}$ relative to a Cartesian frame and coordinates $X^A$ relative to an orthogonal curvilinear frame at time $t_o$. As the deformation progresses, the $Z^{\alpha}$ and $X^A$ coordinates of a material point will change accordingly. Points in the deformed body are assigned

coordinates $z^i$ relative to a second Cartesian system and coordinates $x^i$ relative to a second orthogonal curvilinear frame, at time $t$, $t \neq t_0$. For ease of discussion we shall require that the origins of these four systems lie within the material body, although this is not essential. Having chosen the four reference frames, we now mark those points of the body which lie along the $Z^\alpha$ and $z^i$ axes in the initial and deformed body. Next we find the images in the deformed body of those points lying along the $Z^1$, $Z^2$, and $Z^3$ axes in the original body and label these image lines $\Phi^1$, $\Phi^2$, and $\Phi^3$, respectively. Likewise the image lines in the initial configuration, of those particles lying along the $z^1$, $z^2$, and $z^3$ axes in the deformed body, are to be labeled $\varphi^1$, $\varphi^2$, and $\varphi^3$. Usually neither the $\Phi^\alpha$ nor the $\varphi^\alpha$ lines will be orthogonal. Thus

$$z^\alpha = z^\alpha(Z^\beta, t), \qquad x^i = x^i(z^\alpha) \tag{15}$$

and

$$Z^\alpha = Z^\alpha(X^A), \qquad X^A = X^A(Z^\alpha) \tag{16}$$

while

$$|\varphi^\alpha| = |z^\alpha|, \qquad |\Phi^\alpha| = |Z^\alpha| \tag{17}$$

whenever specific values 1, 2, or 3 are assigned to $\alpha$. In other words, the trio of coordinate numbers associated with a particular particle of the body is the same for both the $\varphi^\alpha$ and $z^\alpha$ frames. This particle will generally be assigned a different trio of coordinate numbers relative to the $Z^\alpha$ frame, but this second trio will agree with the particle's $\Phi^\alpha$ coordinate numbers.

It appears that this choice of coordinates is superior to that in which the deformation is described in terms of the nonorthogonal system comprising the deformed lines of the original frame, as in section 2.1 of reference [4].

This is because the stress tensor may herein be directly related to the displacements. Moreover, it is usually quite difficult to formulate problems in terms of the deformed state when the coordinates to be used are so vague that the metric itself is unknown.

Forgetting for the moment that we are concerned with deformation, we observe that the four coordinate systems $Z^\alpha$, $X^A$, $z^\alpha$, and $x^i$ describe a common space, whose squared element of arc is given by

$$(d\sigma)^2 = G_{MN}\, dX^M\, dX^N = dZ^\alpha\, dZ^\alpha = g_{ij}\, dx^i\, dx^j = dz^\alpha\, dz^\beta \tag{18}$$

where repeated indices signify summation over the range 1, 2, 3. Hence,

$$\left.\begin{aligned} G_{MN} &= g_{ij}\, r^i_M\, r^j_N \\ g_{ij} &= G_{MN}\, R^M_i\, R^N_j \end{aligned}\right\} \tag{19}$$

where

$$r^i_M = x^i_\alpha\, Z^\alpha_M\,, \qquad R^M_i = X^M_\alpha\, z^\alpha_i \tag{20}$$

in terms of the notation

$$X^A_\alpha \equiv \frac{\partial X^A}{\partial z^\alpha}\,, \qquad x^i_\alpha \equiv \frac{\partial x^i}{\partial z^\alpha}\,,$$

$$Z^\alpha_M \equiv \frac{\partial Z^\alpha}{\partial X^M}\,, \qquad z^\alpha_i \equiv \frac{\partial z^\alpha}{\partial x^i}$$

to which we add, for future use, that

$$Z^\alpha_\beta \equiv \frac{\partial Z^\alpha}{\partial z^\beta}\,, \qquad z^\alpha_\beta \equiv \frac{\partial z^\alpha}{\partial Z^\beta}\,,$$

$$\varphi^\alpha_\beta \equiv \frac{\partial \varphi^\alpha}{\partial z^\beta}\,, \qquad \Phi^\alpha_\beta \equiv \frac{\partial \Phi^\alpha}{\partial Z^\beta}\,.$$

Remembering now that we are in fact concerned with the measurement of deformation, we next consider an arc length made up of marked elements within the undeformed material body, for which

$$(dS)^2 = G_{MN}\, dX^M\, dX^N = c_{ij}\, dx^i\, dx^j \tag{21}$$

Similarly, the square of incremental arc length in the deformed body is given by

$$(ds)^2 = g_{ij}\, dx^i\, dx^j = C_{MN}\, dX^M\, dX^N \tag{22}$$

Hence the Green-St. Venant measure of strain, which is referred to the coordinates of the initial configuration, is

$$E_{AB} = \frac{1}{2}\,(C_{AB} - G_{AB}), \tag{23}$$

and the Alamansi-Hamel measure, referred to the coordinates of the deformed body, is

$$\epsilon_{ij} = \frac{1}{2}\,(g_{ij} - c_{ij}) \tag{24}$$

In order to phrase the right-hand sides of (23) and (24) in terms of displacements, we begin by noting that

$$C_{MN} = g_{ij}\, s^i_M\, s^j_N \tag{25}$$

where

$$s^i_M = x^i_\alpha\, \varphi^\alpha_\gamma\, Z^\gamma_M \tag{26}$$

and that

$$c_{ij} = G_{MN}\, S^M_i\, S^N_j \tag{27}$$

where

$$S^M_i = X^M_\alpha\, \Phi^\alpha_\beta\, z^\beta_i\ . \tag{28}$$

Thus

$$E_{AB} = \frac{1}{2}\, G_{MN}\,(H^M_A\, H^N_B - \delta^M_A\, \delta^N_B) \tag{29}$$

and

$$\epsilon_{ij} = \frac{1}{2} g_{mn} \left( \delta_i^m \delta_j^n - h_i^m h_j^n \right) \tag{30}$$

in which

$$H_A^M = R_i^M s_A^i = X_\alpha^M \varphi_\gamma^\alpha Z_A^\gamma \tag{31}$$

and

$$h_i^m = r_A^m S_i^A = x_\alpha^m \Phi_\beta^\alpha z_i^\beta \tag{32}$$

A glance at Fig. 2 shows that the displacement may be found from either

$$U^\alpha = Z_\beta^\alpha z^\beta - \varphi^\alpha \tag{33}$$

in the $Z^\alpha$ frame, or from

$$u^\alpha = \Phi^\alpha - z_\beta^\alpha Z^\beta \tag{34}$$

in the $z^\alpha$ frame. Differentiation of (33) with respect to $Z^\gamma$ reveals that

$$U^\alpha{}_{,\gamma} = \delta_\gamma^\alpha - \varphi_\gamma^\alpha , \tag{35}$$

so that

$$H_A^M = X_\alpha^M \left( \delta_\gamma^\alpha - U^\alpha{}_{,\gamma} \right) Z_A^\gamma \tag{36}$$

$$= \delta_A^M - U^\alpha{}_{,\gamma} Z_A^\gamma X_\alpha^M . \tag{37}$$

Correspondingly, differentiation of (34) with respect to $z^\gamma$ shows that

$$u^\alpha{}_{,\gamma} = \Phi^\alpha{}_{,\gamma} - \delta_\gamma^\alpha . \tag{38}$$

and hence

$$h_i^m = x_\alpha^m u^\alpha{}_{,\beta} z_i^\beta + \delta_\gamma^\alpha \tag{39}$$

Substitution for $H_A^M$ from (36) into (29) and for $h_i^m$ from (39) into (30) results in

$$E_{AB} = \frac{1}{2} Z_A^\gamma Z_B^\eta (U_{\gamma,\eta} + U_{\eta,\gamma} + U_{\lambda,\gamma} U_{\lambda,\eta}) \tag{40}$$

and

$$\epsilon_{ij} = \frac{1}{2} z_i^\beta z_j^\eta (u_{\eta,\beta} + u_{\beta,\eta} - u_{\tau,\beta} u_{\tau,\eta}) \tag{41}$$

Although (40) and (41) are correct as they are written, they express $E_{AB}$ (or $\epsilon_{ij}$) in terms of quantities measured in the $X^A$ and $Z^\alpha$ (or $x^i$ and $z^\alpha$) frames. This use of two reference frames may be reduced to one by noting that

$$U_\alpha = U_\alpha(U^A, X^B, t) \tag{42}$$

and

$$u_\alpha = u_\alpha(u^i, x^i, t) \tag{43}$$

At a <u>particular</u> time $t = t_1$ the displacements are effectively time independent so that

$$U_{\alpha,\beta} = \Xi_{\alpha A} U^A{}_{,B} X_\beta^B + \tilde{U}_{\alpha,A} X_\beta^A \tag{44}$$

$$u_{\alpha,\beta} = \xi_{\alpha i} u^i{}_{,j} x_\beta^j + \tilde{u}_{\alpha,j} x_\beta^j \tag{45}$$

where

$$\Xi_{\alpha A} \equiv \frac{\partial U_\alpha}{\partial U^A}, \qquad \xi_{\alpha i} \equiv \frac{\partial u_\alpha}{\partial u^i}$$

and where

$$\tilde{U}_{\alpha,A} \equiv \frac{\partial U_\alpha}{\partial X^A}, \qquad \tilde{u}_{\alpha,i} \equiv \frac{\partial u_\alpha}{\partial x^i}$$

have been introduced to emphasize that these four terms in (44) and (45) depend upon the transformations as shown in (42) and (43), while $U^A{}_{,B}$ and $u^i{}_{,j}$ terms depend upon the spatial functions obtained from

$$U^A = U^A(X^B, t) \tag{46}$$

$$u^i = u^i(x^j, t) \tag{47}$$

evaluated at time $t = t_1$.

Substitution for the displacement derivatives from (44) and (45) into (40) and (41) results in the desired expressions

$$E_{AB} = \frac{1}{2}\Big[(Z_A^\eta U^C{}_{,B} + Z_B^\eta U^C{}_{,A}) \,\Xi_{\eta C} + Z_A^\eta \tilde{U}_{\eta,B} + Z_B^\eta \tilde{U}_{\eta,A} + (\Xi_{\tau C} U^C{}_{,A} + \tilde{U}_{\tau,A})(\Xi_{\tau D} U^D{}_{,B} + \tilde{U}_{\tau,B})\Big] \tag{48}$$

and

$$\epsilon_{ij} = \frac{1}{2}\Big[(z_i^\eta u^k{}_{,j} + z_j^\eta u^k{}_{,i})\xi_{\eta k} + z_i^\eta \tilde{u}_{\eta,j} + z_j^\eta \tilde{u}_{\eta,i} + (\xi_{\tau k} u^k{}_{,i} + \tilde{u}_{\tau,i})(\xi_{\tau n} u^n{}_{,j} + \tilde{u}_{\tau,j})\Big] \tag{49}$$

Notice once again that <u>if</u> $U_\gamma$ or $u_\eta$ were tensors only then would it be correct to write

$$U_{A|B} = Z_A^\gamma Z_B^\eta U_{\gamma,\eta} \tag{50}$$

and

$$u_{i|j} = z_i^\gamma z_j^\eta u_{\gamma,\eta} \; ; \tag{51}$$

but since neither $U_A$ nor $u_i$ is a tensor, (44) and (45) must be used as they stand. Obviously (44) and (45) agree with (1) and (2) for (i) any displacements whenever $X^A$ and $x^i$ represent Cartesian coordinates, and for (ii) exclusively radial displacements in terms of either circular cylindrical or spherical coordinates.

4. Strain-Displacement Relations in Circular Cylindrical Coordinates. Consider the geometry illustrated in Fig. 1, wherein the Cartesian coordinates $z^\alpha$ are related to the circular cylindrical coordinates $x^i$ as

$$\left.\begin{aligned} z^1 &= x^1 \cos x^2 \\ z^2 &= x^1 \sin x^2 \\ z^3 &= x^3 \end{aligned}\right\} \qquad (52)$$

and where the noninfinitesimal displacement $u_\alpha$, in the Cartesian system is related to the displacement $u^i$ in the circular cylindrical system according to equations (8). From (8) and (52) we find that the nonzero values of $z^\alpha_i$ are

$$z^1_1 = \cos x^2 \qquad z^2_1 = \sin x^2$$

$$z^1_2 = -x^1 \sin x^2 \qquad z^2_2 = x^1 \cos x^2$$

$$z^3_3 = 1,$$

that the nonzero values of $\xi_{\alpha i}$ are

$$\xi_{11} = \cos(x^2 + u^2)$$

$$\xi_{12} = -(x^1 + u^1)\sin(x^2 + u^2)$$

$$\xi_{21} = \sin(x^2 + u^2)$$

$$\xi_{22} = (x^1 + u^1)\cos(x^2 + u^2)$$

$$\xi_{33} = 1,$$

and that the nonzero values of $\tilde{u}_{\alpha,i}$ are

$$\tilde{u}_{1,1} = \cos(x^2 + u^2) - \cos x^2$$

$$\tilde{u}_{1,2} = -x^1[\sin(x^2 + u^2) - \sin x^2] - u^1 \sin(x^2 + u^2)$$

$$\tilde{u}_{2,1} = \sin(x^2 + u^2) - \sin x^2$$

$$\tilde{u}_{2,2} = x^1[\cos(x^2 + u^2) - \cos x^2] + u^1 \cos(x^2 + u^2).$$

Substitution of these values into (49) and subsequent algebraic simplification results in the following strain-displacement relations, expressed in circular cylindrical coordinates:

$$\epsilon_{11} = u^1{}_{,1} (2 \cos u^2 - 1) + 2 (\cos u^2 - 1)$$
$$- \frac{1}{2}[(u^1{}_{,1})^2 + (u^2{}_{,1})^2 (x^1 + u^1)^2 + (u^3{}_{,1})^2]$$

$$\epsilon_{12} = (u^1{}_{,1} - u^2{}_{,2}) x^1 \sin u^2 - u^2{}_{,2} u^1 \sin u^2 + u^1{}_{,2} \cos u^2$$
$$+ u^2{}_{,1} x^1 (x^1 + u^1)\cos u^2 - \frac{1}{2} [u^1 \sin u^2 + u^1{}_{,2} (u^1{}_{,1} + 1)$$
$$+ u^2{}_{,1} (u^2{}_{,2} + 1) (x^1 + u^1) + u^3{}_{,1} u^3{}_{,2}]$$

$$\epsilon_{13} = -u^2{}_{,3} (x^1 + u^1) \sin u^2 + \frac{1}{2} [u^1{}_{,3} (2 \cos u^2 - 1)$$
$$+ u^3{}_{,1} (1 - u^3{}_{,3}) - u^1{}_{,1} u^1{}_{,3} - u^2{}_{,1} u^2{}_{,3} (x^1 + u^1)^2]$$

$$\epsilon_{22} = (x^1 + u^1) [(2 \cos u^2 - 1) x^1 - u^1] u^2{}_{,2} + 2 u^1{}_{,2} x^1 \sin u^2$$
$$+ u^1 x^1 (2 \cos u^2 - 1) + (x^1)^2 (\cos u^2 - 1) - \frac{1}{2} [(u^1{}_{,2})^2$$
$$+ (u^2{}_{,2})^2 (x^1 + u^1)^2 + (u^1)^2 + (u^3{}_{,3})^2]$$

$$\epsilon_{23} = u^1{}_{,3} x^1 \sin u^2 + u^2{}_{,3} x^1 (x^1 + u^1) \cos u^2$$
$$+ \frac{1}{2} [u^3{}_{,2} (1 - u^3{}_{,3}) - u^2{}_{,3} (1 + u^2{}_{,2}) (x^1 + u^1)]$$

$$\epsilon_{33} = u^3{}_{,3} - \frac{1}{2} [(u^1{}_{,3})^2 + (u^2{}_{,3})^2 (x^1 + u^1)^2 + (u^3{}_{,3})^2]$$

In terms of the more common $r, \theta, z$ form of the circular cylindrical coordinates, the physical components of the preceding six equations read

$$\epsilon_{rr} = u_{,r} \left(2 \cos \frac{v}{r} - 1\right) + 2\left(\cos \frac{v}{r} - 1\right)$$
$$- \frac{1}{2}\left[(u_{,r})^2 + \left(1 + \frac{u}{r}\right)^2 \left(v_{,r} + \frac{v}{r}\right)^2 + (w_{,r})^2\right] \qquad (53)$$

$$\epsilon_{\theta\theta} = \left(1 + \frac{u}{r}\right) \left(2 \cos \frac{v}{r} - \frac{u}{r} - 1\right) \frac{v_{,\theta}}{r} + 2 \frac{u_{,\theta}}{r} \sin \frac{v}{r}$$
$$+ \frac{u}{r}\left(2 \cos \frac{v}{r} - 1\right) + \cos \frac{v}{r} - 1 - \frac{1}{2}\left[\left(\frac{u_{,\theta}}{r}\right)^2\right.$$
$$\left. + v_\theta \left(1 + \frac{u}{r}\right)^2 + \left(\frac{u}{r}\right)^2 + (w_{,z})^2\right] \qquad (54)$$

$$\epsilon_{zz} = w,_z - \frac{1}{2}\left[(u,_z)^2 + \left(1 + \frac{u}{r}\right)^2 (v,_z)^2 + (w,_z)^2\right] \tag{55}$$

$$\begin{aligned}\epsilon_{r\theta} &= \left(u,_r - \frac{v,_\theta}{r}\right) \sin\frac{v}{r} - u\,\frac{v,_\theta}{r} \sin\frac{v}{r} + \frac{u,_\theta}{r}\cos\frac{v}{r} \\ &+ \left(v,_r - \frac{v}{r}\right)\left(1 + \frac{u}{r}\right)\cos\frac{v}{r} - \frac{1}{2}\left[\frac{u}{r}\sin\frac{v}{r} + \frac{u,_\theta}{r}(u,_r + 1)\right. \\ &\left. + \left(1 + \frac{u}{r}\right)^2 (v,_\theta + 1)\left(v,_r - \frac{v}{r}\right) + v,_r\, v,_\theta\right]\end{aligned} \tag{56}$$

$$\begin{aligned}\epsilon_{rz} &= -v,_z\,(r + u)\sin\frac{v}{r} + \frac{1}{2}\left[u,_z\left(2\cos\frac{v}{r} - 1\right)\right. \\ &\left. + w,_r\,(1 - w,_z) - u,_r\,u,_z - v,_r\,v,_z\,(r + u)\right]\end{aligned} \tag{57}$$

$$\begin{aligned}\epsilon_{z\theta} &= u,_z \sin\frac{v}{r} + v,_z\left(1 + \frac{u}{r}\right)\cos\frac{v}{r} \\ &+ \frac{1}{2}\left[\frac{w,_\theta}{r}(1 + w,_z) - v,_z\,(1 + v,_\theta)\left(1 + \frac{u}{r}\right)^2\right]\end{aligned} \tag{58}$$

where $u$, $v$, and $w$ represent the displacements along the $r$, $\theta$, $z$ coordinate lines, respectively. It is easy to show that (53) through (58) reduce to the classical infinitesimal strain-displacement relations whenever $u$, $v$, and $w$ are allowed to become so small that second and higher order terms may be neglected.

Strain-displacement relations in the original coordinate system assume the form

$$\begin{aligned}E_{11} &= U^1,_1 + 2\,U^2,_1\,(X^1 + U^1)\sin U^2 \\ &+ \frac{1}{2}\left[(U^1,_1)^2 + (U^2,_1)^2\,(X^1 + U^1)^2 + (U^3,_1)^2\right]\end{aligned} \tag{59}$$

$$E_{22} = U^2{}_{,2} \, (X^1 + U^1)^2 + X^1 \, U^1 + \frac{1}{2} \, [(U^1{}_{,2})^2$$

$$+ (U^2{}_{,2})^2 \, (X^1 + U^1)^2 + (U^1)^2 + (U^3{}_{,2})^2] \qquad (60)$$

$$E_{33} = U^3{}_{,3} + \frac{1}{2} \, [(U^1{}_{,3})^2 + (U^2{}_{,3})^2 \, (X^1 + U^1)^2 + (U^3{}_{,3})^2] \qquad (61)$$

$$E_{12} = \frac{1}{2} \, [U^1{}_{,2} + U^2{}_{,1} \, (X^1 + U^1)^2 - U^1 \sin U^2 + U^1{}_{,1} \, U^1{}_{,2}$$

$$+ U^2{}_{,1} \, U^2{}_{,2} \, (X^1 + U^1)^2 + U^3{}_{,1} \, U^3{}_{,2}] \qquad (62)$$

$$E_{13} = \frac{1}{2} \, [U^3{}_{,1} + U^1{}_{,3} + U^1{}_{,1} \, U^1{}_{,3} + U^2{}_{,1} \, U^2{}_{,3} \, (X^1 + U^1) + U^3{}_{,1} \, U^3{}_{,3}] \qquad (63)$$

$$E_{23} = \frac{1}{2} \, [U^3{}_{,2} + U^2{}_{,3} \, (X^1 + U^1)^2 + U^1{}_{,3} \, U^1{}_{,2} + U^2{}_{,3} \, U^2{}_{,2} \, (X^1 + U^1)^2 + U^3{}_{,3} \, U^3{}_{,2}] \qquad (64)$$

Equations (59) through (64) also reduce to the classical infinitesimal relations whenever second and higher order displacement terms may be neglected.

4. Conclusions. Usually equations derived from (48) and (49) differ from the strain displacements derived from (1) and (2). Acceptance of (1) and (2) in the past may, perhaps, be attributed to their application to problems involving radial displacements exclusively, in which case (1) and (2) agree with (48) and (49), respectively.

~~Equations corresponding to (53) through (58) for other coordinate systems~~ may be obtained directly from equations (48) and (49), which are general tensor equations.

## References

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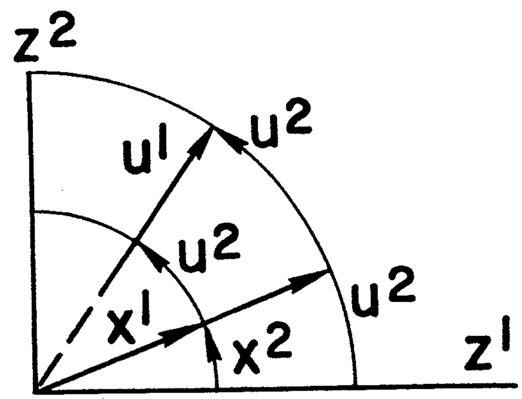
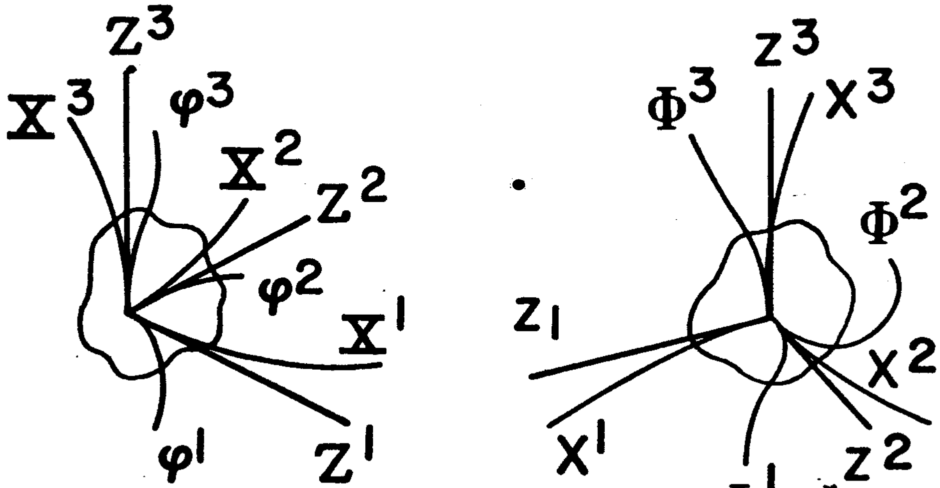



Figure 1.

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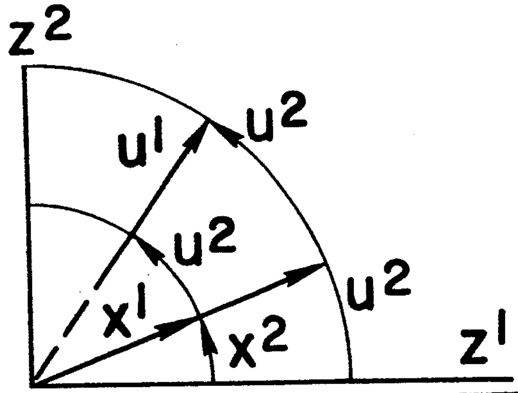


Figure 1.